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ERRORS-IN-VARIABLES FOR MOBILE MAPPING ALGORITHMS IN THE PRESENCE OF OUTLIERS

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ABSTRACT: A few years ago, Schaffrin and Iz (2008) generalized the traditional Kalman filter in such a way that it could handle observation equations with errors-in-variables. This approach led to what has since become known as Total Kalman Filtering (TKF). A drawback, however, was that the usual “data snooping” techniques were no longer applicable in the same manner. Therefore, in the presence of outliers, new search techniques need to be devised in order to accommodate for those errors-in-variables with non-zero expectations. In this contribution, an attempt will be described to prepare a suitable algorithm for this purpose in the context of mobile mapping.

INTRODUCTION

For Mobile Mapping applications that are oftentimes based on the integration of GPS and INS - and possibly further - sensors, algorithms from the Kalman filter family are usually chosen which - after linearization - can be best derived in a Dynamic Linear Model (DLM). More often than not, however, the matrix in the observation equations is also filled with *measured* quantities and must, therefore, be considered as *affected by random errors*. This generalization eventually led to the formulation of the Total Kalman Filter by Schaffrin and Iz (2008).

While the Total Kalman Filter can optimally reduce the influence from all the random errors (EIV), it is certainly not immune against the *occasional outlier* that may occur in any of the measurements, no matter whether they belong to the observation vector or the coefficient matrix. Hence, an attempt to identify/estimate such outliers is in order and will ultimately lead to a “data snooping” procedure similar to the one developed by Baarda (1968) for the simple Gauss-Markov Model.

In the following chapter 1, the standard EIV-Model will be reviewed, and two of the most popular algorithms will be presented that both are able to generate the TLS solution quite efficiently. Afterwards, in chapter 2, the observation vector on the left side is allowed to contain one outlier at a time which needs to be estimated. Moreover, its effect on the other estimated parameters ought to be determined as well; ultimately, a decision must be made as to whether this effect is tolerable or not. Finally, the far more complex case of an outlier in the coefficient matrix is considered in chapter 3, and formulas for its estimation are given, before some conclusions are drawn in chapter 4.

1. A REVIEW OF THE EIV-MODEL

The standard model with Errors-in-Variables (EIV), i.e. with a coefficient matrix that is affected by random errors, can be defined by the observation equation

$$y = \underbrace{(A\text{-}E_A)}_{n\times m}\xi + e\ , \quad n > m = rkA\ , \tag{1.1a}$$

where e denotes the usual $n\times 1$ random error vector and E_A the new $n\times m$ random error matrix, for both of which the stochastic characteristics may be given as

$$\begin{bmatrix} e \\ e_A := vecE_A \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_o^2 \begin{bmatrix} Q & 0 \\ 0 & I_m \otimes Q \end{bmatrix} \right) \tag{1.1b}$$

Here, y denotes the $n\times 1$ vector of (incremental) observations, that are linearly related to the $m\times 1$ vector ξ of unknown parameters through the stochastic (observed) $n\times m$ coefficient matrix A of rank $m < n$.

Note that both e and E_A possess the same (unknown) variance component σ_o^2, are uncorrelated, but involving the same cofactor matrix Q for all columns of A as for the vector y. As usually, "vec" denotes the operation that transforms a matrix into a vector by stacking all its columns one underneath the previous one, and

$\otimes$ denotes the so-called "Kronecker-Zehfuss product" of matrices, defined by

$$G\otimes H := [g_{ij}\cdot H] \ \text{ if } \ G = [g_{ij}]\ ; \tag{1.2}$$

more details about this unconventional product can be found in the Appendix of the textbook by Grafarend and Schaffrin (1993) among many other sources.

Obviously, in spite of the original linearization, the observation equations (1.1a) can be rewritten in the form

$$y = A\xi + \begin{bmatrix} I_n, & -(\xi^T \otimes I_n) \end{bmatrix} \begin{bmatrix} e \\ e_A \end{bmatrix} \tag{1.3}$$

which, along with (1.1b), would form a classical nonlinear Gauss-Helmert Model in the sense of Helmert (1907).

Herein, Least-Squares adjustment could obviously be treated by *iterative model linearization* according to the principles by Pope (1972), as shown by Neitzel and Petrovic (2008). But a more direct approach had been designed by Schaffrin et al. (2006) already that leads to the formulation of *nonlinear normal equations* and avoids a model change altogether. Moreover, it had been shown by Schaffrin (2006; 2007) that this approach is a true generalization of the classical approach by Golub and van Loan (1980) that results in a certain *eigenvalue problem* and is associated with the notion of Total Least-Squares (TLS) estimation. When applying the traditional Lagrange technique, with $P := Q^{-1}$ and λ as $n \times 1$ vector of *Lagrange multipliers*, the following target function

$$\begin{aligned}\Phi(e, e_A, \xi, \lambda) &:= e^T P e + e_A^T (I_m \otimes P) e_A + \\ &+ 2\lambda^T \left[y - e - \left(\xi^T \otimes I_n \right) \left(vecA - e_A \right) \right]\end{aligned} \tag{1.4}$$

ought to be made stationary in order to find the TLS solution. The necessary conditions are easily derived by forming the respective partial derivatives and setting them to zero. This leads to the following *Euler-Lagrange conditions*:

$$\frac{1}{2}\frac{\partial \Phi}{\partial e} = P\tilde{e} - \hat{\lambda} \doteq 0 \qquad \Rightarrow \boxed{\tilde{e} = Q\hat{\lambda}} \tag{1.5a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_A} = \left(I_m \otimes P \right)\tilde{e}_A + \left(\hat{\xi} \otimes I_n \right)\hat{\lambda} \doteq 0 \qquad \Rightarrow \qquad \Rightarrow \quad \tilde{e}_A = -\left(\hat{\xi} \otimes Q \right)\hat{\lambda} \qquad \Rightarrow \boxed{\tilde{E}_A = -Q\hat{\lambda}\hat{\xi}^T}$$

(1.5b)

$$\frac{1}{2}\frac{\partial \Phi}{\partial \xi} = -\left(A - \tilde{E}_A \right)^T \hat{\lambda} \doteq 0 \qquad \Rightarrow \qquad \boxed{A^T\hat{\lambda} = \tilde{E}_A^T\hat{\lambda} = -\hat{\xi} \cdot \left(\hat{\lambda}^T Q \hat{\lambda} \right)} \tag{1.5c}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \lambda} = y - \tilde{e} - A\hat{\xi} + \tilde{E}_A\hat{\xi} \doteq 0 \qquad \Rightarrow \qquad \boxed{P(y - A\hat{\xi})(1 + \hat{\xi}^T\hat{\xi})^{-1} = \hat{\lambda}} \tag{1.5d}$$

as, from (1.5a-b), the following relationships can be derived:

$$P(y - A\hat{\xi}) = P(\tilde{e} - \tilde{E}_A\hat{\xi}) = \hat{\lambda}(1 + \hat{\xi}^T\hat{\xi}) \tag{1.6a}$$

$$\Rightarrow \quad A^T\hat{\lambda} = (c - N\hat{\xi})(1 + \hat{\xi}^T\hat{\xi})^{-1} \text{ if } \quad [c, N] := A^T P [y, A] \tag{1.6b}$$

Thus, from (1.5c-d) and (1.6b), the following *normal equations* follow now immediately:

$$c - N\hat{\xi} = A^T\hat{\lambda}(1 + \hat{\xi}^T\hat{\xi}) = -\hat{\xi} \cdot \hat{v} \Rightarrow \boxed{(N - \hat{v}I_m)\hat{\xi} = c} \tag{1.7a}$$

with

$$\boxed{\begin{aligned}\hat{v} = (\hat{\lambda}^T Q \hat{\lambda})(1 + \hat{\xi}^T\hat{\xi}) &= \tilde{e}^T P \tilde{e} + \tilde{e}_A^T (I_n \otimes P)\tilde{e}_A = \\ = (1 + \hat{\xi}^T\hat{\xi})^{-1}&(y - A\hat{\xi})^T P (y - A\hat{\xi}) = TSSR\end{aligned}} \tag{1.7b}$$

as "Total Sum of Squared Residuals". By neglecting the randomness of $\hat{v}$, a first-order approximation for the dispersion matrix can now be obtained as:

$$\boxed{\begin{aligned} D\{\hat{\xi}\} &\approx \sigma_o^2 (N - \hat{v} I_m)^{-1} N (N - \hat{v} I_m)^{-1} = \\ &= \sigma_o^2 (N - \hat{v} I_m)^{-1} + \sigma_o^2 \hat{v} (N - \hat{v} I_m)^{-2} \end{aligned}} \tag{1.8}$$

In addition, (1.5d) can be transformed into the relationship:

$$\begin{aligned} (y^T P y - c^T \hat{\xi})(1 + \hat{\xi}^T \hat{\xi})^{-1} &= y^T \hat{\lambda} = \\ &= (y - A\hat{\xi})^T P \cdot Q\hat{\lambda} + \hat{\xi}^T \cdot A^T \hat{\lambda} = \\ &= (1 + \hat{\xi}^T \hat{\xi}) \cdot (\hat{\lambda}^T Q \hat{\lambda}) - \hat{\xi}^T \cdot \hat{\xi} (\hat{\lambda}^T Q \hat{\lambda}) = \hat{\lambda}^T Q \hat{\lambda} \end{aligned} \tag{1.9a}$$

or, equivalently, into:

$$\boxed{y^T P y - c^T \hat{\xi} = (1 + \hat{\xi}^T \hat{\xi}) \cdot (\hat{\lambda}^T Q \hat{\lambda}) = \hat{v}} \tag{1.9b}$$

Combining now (1.9b) with (1.7a) leads to

$$\begin{bmatrix} N & c \\ c^T & y^T P y \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} = \hat{v} \cdot \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix}, \quad \hat{v} = v_{\min}, \tag{1.10}$$

which shows that $\hat{v}$ turns indeed out as the *minimum eigenvalue* of the matrix on the left side of equation (1.10). This obviously was the original contribution of Golub and van Loan (1980).

In the following two chapters, the aim will be directed towards generalizing the formulas above to the two cases where either the vector y or the matrix A is, in a single component, affected by an outlier.

2. OUTLIERS AFFECTING THE OBSERVATION VECTOR

In this chapter it is assumed that *one outlier* may have occurred in the observation vector y, and none in the coefficient matrix A, thus leading to the *modified* observation equations

$$y = \underset{n \times m}{A} \xi + \underset{n \times 1}{\eta_j} \xi_o^{(j)} + (e - \underset{n \times m}{E_A} \xi) \tag{2.1a}$$

where η_j denotes the j-th unit vector (here of size $n \times 1$). Then, the stochastic characteristics for e and e_A can be maintained as given in formula (1.1b).

Consequently, the corresponding Lagrange approach as sketched in chapter 1 will have to be modified accordingly, thus starting from the target function

$$\begin{aligned}\Phi(e, e_A, \xi, \xi_o^{(j)}, \lambda) &:= e^T P e + e_A^T (I_m \otimes P) e_A + \\ &+ 2\lambda^T \left[y - e - \left(\xi^T \otimes I_n\right)\left(vecA - e_A\right) - \eta_j \xi_o^{(j)} \right]\end{aligned} \tag{2.1b}$$

before making it stationary for the identification of the new TLS solution. This leads to the three *Euler-Lagrange conditions* (1.5a-c) and two new ones, namely:

$$\frac{1}{2}\frac{\partial \Phi}{\partial \xi_o^{(j)}} = -\eta_j^T \hat{\lambda}^{(j)} \doteq 0 \qquad \Rightarrow \boxed{\hat{\lambda}_j^{(j)} = 0} \tag{2.2a}$$

$$\Rightarrow \tilde{e}_j^{(j)} = 0 \quad \text{if } Q \text{ is diagonal;}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial \Phi}{\partial \lambda} &= y - \tilde{e}^{(j)} - A\hat{\xi}^{(j)} + \tilde{E}_A^{(j)}\hat{\xi}^{(j)} - \eta_j \hat{\xi}_o^{(j)} \doteq 0 \quad \Rightarrow \\ \Rightarrow P(y - A\hat{\xi}^{(j)} - \eta_j \hat{\xi}_o^{(j)}) &= P\tilde{e}^{(j)} - P\tilde{E}_A^{(j)}\hat{\xi}^{(j)} = \\ &= \hat{\lambda}^{(j)}[1 + (\hat{\xi}^{(j)})^T \hat{\xi}^{(j)}]\end{aligned} \tag{2.2b}$$

Obviously, (2.2b) can be used to solve for

$$\boxed{\hat{\lambda}^{(j)} = P(y - A\hat{\xi}^{(j)} - \eta_j \hat{\xi}_o^{(j)}) \cdot [1 + (\hat{\xi}^{(j)})^T \hat{\xi}^{(j)}]^{-1}} \Rightarrow \tag{2.3a}$$

$$\Rightarrow A^T \hat{\lambda}^{(j)} \cdot [1 + (\hat{\xi}^{(j)})^T \hat{\xi}^{(j)}] = c - N\hat{\xi}^{(j)} - A^T P \eta_j \hat{\xi}_o^{(j)} \tag{2.3b}$$

which becomes one part of the *new normal equations*, using (1.5c), via

$$c - N\hat{\xi}^{(j)} - A^T P \eta_j \hat{\xi}_o^{(j)} = -\hat{\xi}^{(j)} \cdot \hat{v}^{(j)} \quad \Rightarrow \tag{2.3c}$$

$$\Rightarrow \boxed{(N - \hat{v}^{(j)} I_m)\hat{\xi}^{(j)} + A^T P \eta_j \cdot \hat{\xi}_o^{(j)} = c} \tag{2.4a}$$

with

$$\boxed{\begin{aligned}\hat{v}^{(j)} &= [(\hat{\lambda}^{(j)})^T Q \hat{\lambda}^{(j)}] \cdot [1 + (\hat{\xi}^{(j)})^T \hat{\xi}^{(j)}] = \\ &= (\tilde{e}^{(j)})^T P \tilde{e}^{(j)} + (\tilde{e}_A^{(j)})^T (I_m \otimes P) \tilde{e}_A^{(j)} = \\ &= [1 + (\hat{\xi}^{(j)})^T \hat{\xi}^{(j)}]^{-1} \cdot \\ &\quad \cdot (y - A\hat{\xi}^{(j)} - \eta_j \hat{\xi}_o^{(j)})^T P (y - A\hat{\xi}^{(j)} - \eta_j \hat{\xi}_o^{(j)})\end{aligned}} \tag{2.4b}$$

On the other hand, (2.2a) in combination with (2.3a) yields

$$\boxed{\eta_j^T PA\hat{\xi}^{(j)} + (\eta_j^T P\eta_j)\cdot\hat{\xi}_o^{(j)} = \eta_j^T Py} \tag{2.4c}$$

as the second part of the normal equations. (2.4a-c) can be employed to solve for the *estimated size of the outlier* via

$$\begin{aligned}\hat{\xi}_o^{(j)} &= [\eta_j^T(P - PA(N - \hat{\nu}^{(j)} I_m)^{-1} A^T P)\eta_j]^{-1}\cdot \\ &\cdot [\eta_j^T(P - PA(N - \hat{\nu}^{(j)} I_m)^{-1} A^T P)y]\end{aligned} \tag{2.5a}$$

and, by combining (2.4a) with (1.7a), an *update formula* can be provided through

$$\hat{\xi}^{(j)} = \hat{\xi} - (N - \hat{\nu}^{(j)} I_m)^{-1}[A^T P\eta_j \cdot \hat{\xi}_o^{(j)} + (\hat{\nu} - \hat{\nu}^{(j)})\cdot\hat{\xi}], \tag{2.5b}$$

thus leading to the combined system for both, the update $(\hat{\xi}^{(j)} - \hat{\xi})$ and the estimated size $\hat{\xi}_o^{(j)}$,

$$\boxed{\begin{bmatrix} N - \hat{\nu}^{(j)} I_m & A^T P\eta_j \\ \eta_j^T PA & \eta_j^T P\eta_j \end{bmatrix}\begin{bmatrix} \hat{\xi}^{(j)} - \hat{\xi} \\ \hat{\xi}_o^{(j)} \end{bmatrix} = \begin{bmatrix} -\hat{\xi}(\hat{\nu} - \hat{\nu}^{(j)}) \\ \eta_j^T P(y - A\hat{\xi}) \end{bmatrix}} \tag{2.6a}$$

where

$$\boxed{y - A\hat{\xi} = \tilde{e} - \tilde{E}_A\hat{\xi} = \tilde{e}(1 + \hat{\xi}^T\hat{\xi})} \tag{2.6b}$$

represents the *residual part* of the *original* TLS solution without outliers, while $(\hat{\nu} - \hat{\nu}^{(j)})$ measures the increase in the TSSR due to the neglected outlier. In a first-order approximation, it may hence be conjectured that the *test statistic*

$$T_j := \frac{\hat{\nu} - \hat{\nu}^{(j)}}{\hat{\nu}^{(j)}/(n - m - 1)} \sim F(1, n - m - 1) \tag{2.7}$$

follows a central F-distribution under the *null hypothesis* that "no outlier is present" in the j-th observation, $y_j = \eta_j^T y$.

Unfortunately, due to the nature of (2.4c), it does not seem possible to replace the above formulas in such a way that an equivalent eigenvalue problem arises where $\hat{\nu}^{(j)}$ would now represent the modified smallest eigenvalue. So, we leave this issue as an open question, and rather turn to the second case where the coefficient matrix may contain one single outlier.

3. OUTLIERS AFFECTING THE COEFFICIENT MATRIX

Now, in order to take an outlier in the coefficient matrix A into account, the original observation equation (1.1a) is modified as follows:

$$y = \underset{n\times m}{A}\,\xi - \underbrace{(\eta_j \xi_o^{(jk)} \eta_k^T)}_{n\times m} \cdot \xi + (e - \underset{n\times m}{E_A}\,\xi) \tag{3.1a}$$

where η_j denotes the j-th unit vector of size $n\times 1$ and η_k the k-th unit vector of size $m\times 1$. With this modification, the stochastic characteristics of e and e_A may be maintained as given in formula (1.1b).

Following the *Lagrange approach* as before, the relevant target function now reads:

$$\begin{aligned}\Phi(e, e_A, \xi, \xi_o^{(jk)}, \lambda) &:= e^T P e + e_A^T (I_m \otimes P) e_A + \\ &+ 2\lambda^T \cdot [y - e - (\xi^T \otimes I_n)(vecA - e_A - (\eta_k \otimes \eta_j)\xi_o^{(jk)}]\end{aligned} \tag{3.1b}$$

which needs to be made stationary in order to find the new TLS solution. Again, the first two of the necessary *Euler-Lagrange conditions* remain identical to (1.5a-b), but now augmented by three new ones, namely:

$$\frac{1}{2}\frac{\partial\Phi}{\partial\xi} = -A^T \hat{\lambda}^{(jk)} + (\tilde{E}_A^{(jk)})^T \hat{\lambda}^{(jk)} + \eta_k \hat{\xi}_o^{(jk)} \eta_j^T \cdot \hat{\lambda}^{(jk)} \doteq 0 \quad \Rightarrow$$

$$\boxed{A^T \hat{\lambda}^{(jk)} = -\hat{\xi}^{(jk)}[(\hat{\lambda}^{(jk)})^T Q \hat{\lambda}^{(jk)}] + \eta_k \cdot \hat{\xi}_o^{(jk)} \cdot (\eta_j^T \hat{\lambda}^{(jk)})} \tag{3.2a}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\xi_o} = (\eta_k^T \otimes \eta_j^T)(\hat{\xi}^{(jk)} \otimes I_n)\hat{\lambda}^{(jk)} \doteq 0 \Rightarrow \boxed{(\eta_k^T \hat{\xi}^{(jk)}) \cdot (\eta_j^T \hat{\lambda}^{(jk)}) = 0} \tag{3.2b}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial\Phi}{\partial\lambda} &= y - \tilde{e}^{(jk)} - A\hat{\xi}^{(jk)} + \tilde{E}_A^{(jk)}\hat{\xi}^{(jk)} + (\eta_j \hat{\xi}_o^{(jk)})(\eta_k^T \hat{\xi}^{(jk)}) \doteq 0 \\ &\Rightarrow P[y - A\hat{\xi}^{(jk)} + (\eta_j \hat{\xi}_o^{(jk)} \eta_k^T)\hat{\xi}^{(jk)}] = \\ &= P\tilde{e}^{(jk)} - P\tilde{E}_A^{(jk)}\hat{\xi}^{(jk)} = \hat{\lambda}^{(jk)} \cdot [1 + (\hat{\xi}^{(jk)})^T \cdot \hat{\xi}^{(jk)}]\end{aligned} \tag{3.2c}$$

(3.2c) can first be used to solve for

$$\boxed{\hat{\lambda}^{(jk)} = P[y - A\hat{\xi}^{(jk)} - \eta_j \hat{\xi}_o^{(jk)} \cdot (\eta_k^T \hat{\xi}^{(jk)})] \cdot [1 + (\hat{\xi}^{(jk)})^T \hat{\xi}^{(jk)}]^{-1}} \tag{3.3a}$$

$$\Rightarrow A^T \hat{\lambda}^{(jk)}[1 + (\hat{\xi}^{(jk)})^T \hat{\xi}^{(jk)}] = c - N\hat{\xi}^{(jk)} - A^T P\eta_j \cdot \xi_o^{(jk)}(\eta_k^T \hat{\xi}^{(jk)}) \tag{3.3b}$$

Furthermore, in (3.2b) it may be assumed without restricting the generality that

$$\eta_k^T \hat{\xi}^{(jk)} = \hat{\xi}_k^{(jk)} \neq 0 \Rightarrow \boxed{\eta_j^T \hat{\lambda}^{(jk)} = \hat{\lambda}_j^{(jk)} = 0} \tag{3.3c}$$

$$\Rightarrow \tilde{e}_j^{(jk)} = 0 \text{ if } Q \text{ is diagonal.}$$

Hence, by combining (3.3b-c) with (3.2a), the first part of the *new normal equations* is obtained as

$$c - N\hat{\xi}^{(jk)} - A^T P\eta_j \cdot (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)}) = -\hat{\xi}^{(jk)} \cdot \hat{v}^{(jk)} + 0 \tag{3.3d}$$

$$\Rightarrow \boxed{(N - \hat{v}^{(jk)} I_m)\hat{\xi}^{(jk)} + A^T P\eta_j (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)}) = c} \tag{3.4a}$$

with

$$\boxed{\begin{aligned} \hat{v}^{(jk)} &= [(\hat{\lambda}^{(jk)})^T Q \hat{\lambda}^{(jk)}] \cdot [1 + (\hat{\xi}^{(jk)})^T \hat{\xi}^{(jk)}] = \\ &= (\tilde{e}^{(jk)})^T P \tilde{e}^{(jk)} + (\tilde{e}_A^{(jk)})^T (I_m \otimes P)\tilde{e}_A^{(jk)} = \\ &= [1 + (\hat{\xi}^{(jk)})^T \hat{\xi}^{(jk)}]^{-1} \cdot \\ &[y - A\hat{\xi}^{(jk)} - \eta_j (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)})]^T P[y - A\hat{\xi}^{(jk)} - \eta_j (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)})] \end{aligned}} \tag{3.4b}$$

In addition, combining (3.3c) with (3.2c) yields the second part via

$$\boxed{\eta_j^T PA \cdot \hat{\xi}^{(jk)} + (\eta_j^T P\eta_j) \cdot (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)}) = \eta_j^T Py} \tag{3.4c}$$

Apparently, these normal equations will be solved for $\hat{\xi}^{(jk)}$ and $(\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)})$, followed by the trivial solution for the *size of the outlier*:

$$\hat{\xi}_o^{(jk)} = (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)}) / (\eta_k^T \hat{\xi}^{(jk)}) \tag{3.4d}$$

In analogy to the approach of chapter 2, an update solution for $(\hat{\xi}^{(jk)} - \hat{\xi})$ - in addition to $(\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)})$ - may be favored in which case (1.7a) is employed to replace the vector c in (3.4a), eventually providing the system

$$\boxed{\begin{bmatrix} N - \hat{\nu}^{(jk)} I_m & A^T P \eta_j \\ \eta_j^T PA & \eta_j^T P \eta_j \end{bmatrix} \begin{bmatrix} \hat{\xi}^{(jk)} - \hat{\xi} \\ \hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)} \end{bmatrix} = \begin{bmatrix} -\hat{\xi}(\hat{\nu} - \hat{\nu}^{(jk)}) \\ \eta_j^T P(y - A\hat{\xi}) \end{bmatrix}} \tag{3.5}$$

where $(y - A\hat{\xi})$ again represents the *residual part* of the *original* TLS solution according to (2.6b). The (scaled) size of the outlier can, therefore, be readily estimated from the original solution via

$$\begin{aligned} (\hat{\xi}_o^{(jk)} \cdot \hat{\xi}_k^{(jk)}) = {} & [\eta_j^T (P - PA(N - \hat{\nu}^{(jk)} I_m)^{-1} A^T P)\eta_j]^{-1} \cdot \\ & \cdot [\eta_j^T P(\tilde{e} + A(N - \hat{\nu}^{(jk)} I_m)^{-1} \hat{\xi}(\hat{\nu} - \hat{\nu}^{(jk)})] \end{aligned} \tag{3.6}$$

once the reduction $(\hat{\nu} - \hat{\nu}^{(jk)})$ in the TSSR can be calculated *directly*, i.e. without referring to the formula (3.4b) itself. This, however, is beyond the scope of the present paper.

Instead, it is again conjectured that, in a first-order approximation, the *test statistic*

$$T_{jk} := \frac{\hat{\nu} - \hat{\nu}^{(jk)}}{\hat{\nu}^{(jk)} / (n - m - 1)} \sim F(1, n - m - 1) \tag{3.7}$$

follows a central F-distribution under the *null hypothesis* that "no outliers are present" in the (jk)-element of the coefficient matrix *A*.

4. CONCLUSIONS AND OUTLOOK

In the case where the observation equations follow an EIV-Model (rather than the standard Gauss-Markov Model), the treatment of a single outlier in either the observation vector or the (observed) coefficient matrix has successfully been completed. The proposed test procedures, however, supposed to establish the significance of any outlier, deserve further attention in regard of their actual probability distribution; the test statistics are presently conjectured to follow a F-distribution in a first-order approximation. On this basis, "data snooping" in the sense of Baarda (1968) is now an option even for EIV-Models; see the small example in the Appendix.

Further open questions are still concerned with the notion of *reliability*, both locally and globally. The respective measures may only depend on the elements provided by the model definition, not by the chosen estimator. This turns out to be particularly tricky in the case of an EIV-Model and has been tackled, at least to some extent, by Schaffrin and Uzun (2011).

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APPENDIX: A simple example - The straight-line adjustment of four points in 2-D (Wolf & Ghilani, 1997, p.426)

Given:
$$\underset{4\times1}{y} = [4.5;\ 4.25;\ 5.5;\ 5.5]^T \ ,$$
$$\underset{4\times2}{A} = \begin{bmatrix} 3.0 & 4.25 & 5.5 & 8.0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^T \ ,$$
$$\underset{2\times1}{\xi} = [\xi_1(\text{slope}); \xi_2(\text{intercept})]^T \ ,$$

such that
$$e = y - (A - E_A)\xi = \ ,$$
$$= (y - A\xi) + (\xi^T \otimes I_4) vec E_A \ ,$$

with:
$$\begin{bmatrix} e \\ e_A \end{bmatrix} \sim \text{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} I_4 & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_4 \end{bmatrix}) \ .$$

Then:
$$\hat{\xi}_{TLS} = [0.2540; 3.6201]^T \ ,$$
$$\hat{v} = 0.4437 = TSSR \ ,$$
$$(\hat{\sigma}_0^2)_{TLS} = \hat{v}/(4-2) = 0.2219 \ .$$

In case of a potential outlier in an individual ordinate y_j (j=1,..,4), the new $(TSSR)_j = \hat{v}^{(j)}$ are listed along with the respective *test statistics* T_j :

	j=1	j=2	j=3	j=4
$\hat{v}^{(j)}$	0.4109	0.1602	0.1460	0.3183
T_j	0.0799	1.7701	2.0393	0.3940

which are all supposed to be F(1,1) distributed; as a result, none of the ordinates are flagged as outliers at the $\alpha = 0.01$ significance level.

In the matrix A, only the abscissae x_j (j=1,..,4) in the first column may be affected by a potential outlier. Again, the new $(TSSR)_{j1} = \hat{v}^{(j1)}$ are listed along with the respective *test statistics* T_{j1} :

	j=1	j=2	j=3	j=4
$\hat{v}^{(j1)}$	0.4109	0.1602	0.1460	0.3183
T_{j1}	0.0799	1.7701	2.0393	0.3940

leading to no flagging of any abscissa either as, due to the symmetry, all the corresponding quantities turn out *identical*.